- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is *BIBO stable* if and only if the ROC of its system function includes the (entire) *unit circle* (i.e., $|z| = $.(1
- **Theorem.** A *causal* LTI system with a *rational* system function H is BIBO stable if and only if all of the poles of H lie inside the unit circle (i.e., each of the poles has a *magnitude less than one*.(

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(z)H_{\text{inv}}(z) = 1,$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(z) = \frac{1}{H(z)}.$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is *not necessarily unique*.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in *one specific choice* of inverse system (due to these additional constraints of stability and/or causality).

System-Function and Difference-Equation Representations of LTI Systems

- Many LTI systems of practical interest can be represented using an *Nth-order linear difference equation with constant coefficients*.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k) \quad \text{where} \quad M \leq N.$$

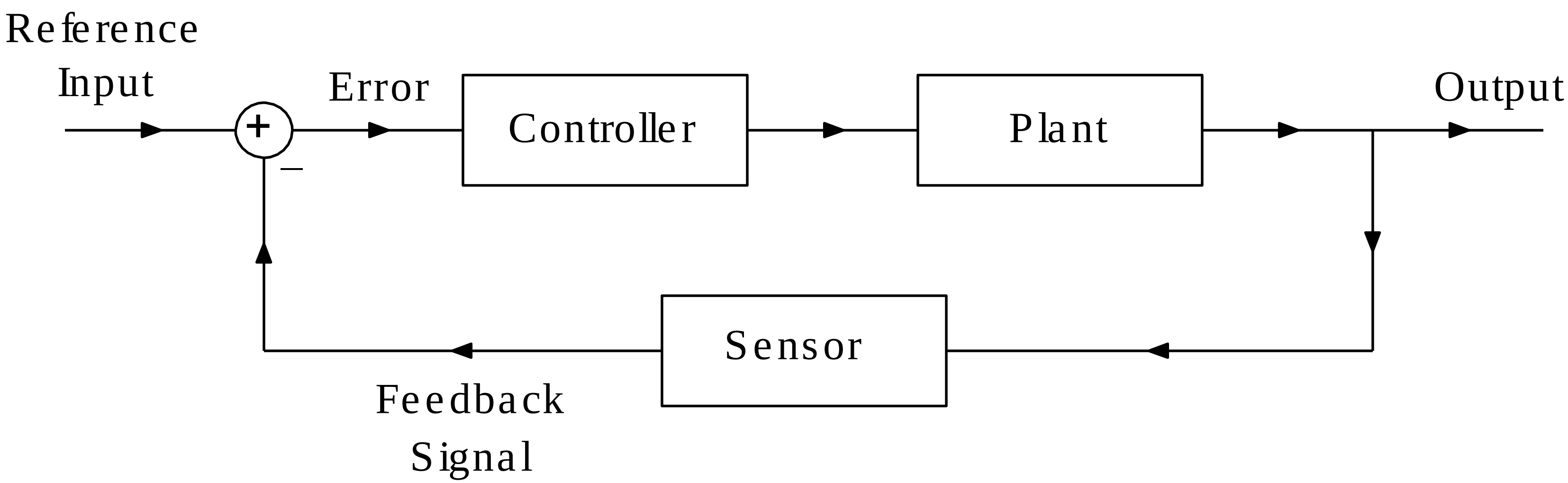
- Let h denote the impulse response of the system, and let X, Y, and H denote the z transforms of x, y, and h, respectively.
- One can show that $H(z)$ is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}.$$

- Observe that, for a system of the form considered above, the system function is always *rational*.

Section 11.6

Application: Analysis of Control Systems



- input: *desired value* of the quantity to be controlled
- output: *actual value* of the quantity to be controlled
- error: *difference* between the desired and actual values
- plant: system to be controlled
- sensor: device used to measure the actual output
- controller: device that monitors the error and changes the input of the plant with the goal of forcing the error to zero

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the z domain than in the time domain.
- Therefore, the z domain is extremely useful for the stability analysis of systems.

Section 11.7

Unilateral Z Transform

- The unilateral z transform of the sequence x, denoted $\mathcal{UZ}\{x\}$ or X, is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

- The unilateral z transform is related to the bilateral z transform as follows:

$$\mathcal{UZ}\{x\}(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) u(n) z^{-n} = \mathcal{Z}\{xu\}(z).$$

- In other words, the unilateral z transform of the sequence x is simply the bilateral z transform of the sequence xu.
- Since $\mathcal{UZ}\{x\} = \mathcal{Z}\{xu\}$ and xu is always a *right-sided* sequence, the ROC associated with $\mathcal{UZ}\{x\}$ is always the *exterior of a circle*.
- For this reason, we often *do not explicitly indicate the ROC* when working with the unilateral z transform.

- With the unilateral z transform, the same inverse transform equation is used as in the bilateral case.
- The unilateral z transform is *only invertible for causal sequences*. In particular, we have

$$\begin{aligned} \mathcal{UZ}^{-1}\{\mathcal{UZ}\{x\}\}(n) &= \mathcal{UZ}^{-1}\{\mathcal{Z}\{xu\}\}(n) \\ &= \mathcal{Z}^{-1}\{\mathcal{Z}\{xu\}\}(n) \\ &= x(n)u(n) \\ &= \begin{cases} x(n) & \text{if } n \geq 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- For a noncausal sequence x, we can only recover $x(n)$ for $n \geq 0$.

- Due to the close relationship between the unilateral and bilateral z transforms, these two transforms have some similarities in their properties.
- Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways.

Property	Time Domain	Z Domain
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$
Time Delay	$x(n-1)$	$z^{-1} X(z) + x(-1)$
Time Advance	$x(n+1)$	$zX(z) - zx(0)$
Z-Domain Scaling	$a^n x(n)$	$X(a^{-1} z)$
	$e^{j\Omega_0 n} x(n)$	$X(e^{-j\Omega_0} z)$
Upsampling	$(\uparrow M) x(n)$	$X(z^M)$
Conjugation	$x^*(n)$	$X^*(z^*)$
Convolution	$x_1 * x_2(n)$, x_1 and x_2 are causal	$X_1(z) X_2(z)$
Z-Domain Diff.	$nx(n)$	$-z\frac{d}{dz}X(z)$
Differencing	$x(n) - x(n-1)$	$(1-z^{-1})X(z) - x(-1)$
Accumulation	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}} X(z)$

Property	
Initial Value Theorem	$x(0) = \lim_{z\to\infty} X(z)$
Final Value Theorem	$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)X(z)$

Pair	$x(n),\ n \geq 0$	$X(z)$
1	$\delta(n)$	1
2	1	$\frac{z}{z-1}$
3	n	$\frac{z}{(z-1)^2}$
4	a^n	$\frac{z}{z-a}$
5	$a^n n$	$\frac{az}{(z-a)^2}$
6	$\cos \Omega_0 n$	$\frac{z(z-\cos\Omega_0)}{z^2-2(\cos\Omega_0)z+1}$
7	$\sin \Omega_0 n$	$\frac{z\sin\Omega_0}{z^2-2(\cos\Omega_0)z+1}$
8	$\lvert a\rvert^n \cos \Omega_0 n$	$\frac{z(z-\lvert a\rvert\cos\Omega_0)}{z^2-2\lvert a\rvert(\cos\Omega_0)z+\lvert a\rvert^2}$
9	$\lvert a\rvert^n \sin \Omega_0 n$	$\frac{z\lvert a\rvert\sin\Omega_0}{z^2-2\lvert a\rvert(\cos\Omega_0)z+\lvert a\rvert^2}$

Solving Difference Equations Using the Unilateral Z Transform

- Many systems of interest in engineering applications can be characterized by constant-coefficient linear difference equations.
- One common use of the unilateral z transform is in solving constant-coefficient linear difference equations with nonzero initial conditions.

Part 12

Complex Analysis

- A complex number is a number of the form $z = x + jy$ where x and y are real numbers and j is the constant defined by $j^2 = -1$ (i.e., $j = \sqrt{-1}$).
- The Cartesian form of the complex number z expresses z in the form

$$z = x + jy,$$

where x and y are real numbers. The quantities x and y are called the real part and imaginary part of z, and are denoted as $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively.

- The polar form of the complex number z expresses z in the form

$$z = r(\cos\theta + j\sin\theta) \quad \text{or equivalently} \quad z = re^{j\theta},$$

where r and θ are real numbers and $r \geq 0$. The quantities r and θ are called the magnitude and argument of z, and are denoted as $|z|$ and $\arg z$, respectively. [Note: $e^{j\theta} = \cos\theta + j\sin\theta$.]